

The General Relativity of Two Properties

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We demonstrate that the extension of the space-time metric to incorporate two anti-commuting property coordinates automatically leads to the unification of gravity with nonabelian gauge theory, as well as producing a cosmological term.

1. Introduction

Space-time describes a ‘when’ and a ‘where’ of an event but not a ‘what’. The traditional approach for addressing the character of an event is to introduce various quantum fields with associated quantum numbers and enforce conservation laws on the combination of quantum numbers. While this approach most certainly works, it leaves us with many open questions, such as why multiple generations of particles exist or why they have their respective masses. Over a number of years we have attempted to introduce some mathematics into the *attributes* of an event by attaching property coordinates for characterizing the ‘what’. These property coordinates are to be associated with Lorentz scalar anti-commuting quantities carrying quantum numbers; the conservation laws for events arise simply by Grassmann integration over the property coordinates in much the same way that overall momentum conservation comes about by integrating over space-time in local field theory. The property coordinates provide a mechanism for explaining why there are repeated generations of particles and they potentially answer^{1,2,3,4,5} other questions left open in the Standard Model.

In a previous paper⁶, we discussed and formalized the general extension of space-time which comes about by attaching multiple property coordinates; this led us to General Relativity (GR) on a graded manifold. We later confined ourselves to a single property coordinate and constructed a supermetric that transformed correctly under local gauge transformations. Using this supermetric we determined the Ricci supertensor and Ricci superscalar and discovered that the Einstein Field equations for gravity plus electromagnetism ensued as well as a cosmological constant. This paper will be an extension of that work, looking at the non-abelian case with two property coordinates. and we shall see that this leads to a unification of gravity with Yang-Mills theory whereby the stress-tensor for the non-abelian gauge fields arises as part of the generalized Ricci tensor.

2. Property coordinates, notation and supermetric

Following our earlier work we will attach to space-time two complex anti-commuting coordinates ζ and their conjugates $\bar{\zeta}$. We choose these coordinates to be Lorentz scalar, which distinguishes our scheme from supersymmetry (where the extra degrees of freedom transform as spinors). The coordinates are anti-commuting so we also avoid infinite particle states, because series expansions in the coordinates will terminate. Quantum numbers are attributed to these property coordinates and particle fields are produced via superfield expansions. Consequences of such a scheme with five property coordinates are covered by papers^{3,4,5}. In this paper we will be focussed purely on the geometrical aspect, namely what happens to general relativity when two of these property coordinates are included.

As before, we adopt the convention that the extended space-time-property indices run over uppercase Roman characters (M, N, L , etc), space-time indices run over lowercase Roman characters (m, n, l , etc) and property indices run over Greek letters (μ, ν, λ , etc). The grading of an index is given by $[M]$, where $[m] = 0$ for even indices and $[\mu] = 1$ for odd indices.

We intend to model our notation on standard GR as far as possible, but this does produce some tension with particle physics notation. Consider a property coordinate ζ^μ ; the conjugate of this coordinate in usual particle physics parlance would be written $\bar{\zeta}_\mu$. On the other hand, raising and lowering indices in GR corresponds to swapping between contravariant and covariant coordinates. Since we are introducing Lorentz *scalar* property coordinates and want to maintain GR notation, we will adopt the following algebraic convention: $\zeta^\mu = \zeta_{\bar{\mu}}$ and $\zeta^{\bar{\mu}} = -\zeta_\mu$. The minus sign exists to ensure self-consistency of the up then down summation of our graded GR with the bar and no bar summation of particle physics; it will also guarantee that we sum barred indices with unbarred ones when they both occur as superscripts or subscripts, in order to produce internal symmetry invariants.

To construct our metric we first consider a flat manifold possessing coordinates $X^M = (x^m, \zeta^\mu, \zeta^{\bar{\mu}})$ without space-time curvature or gauge fields. Our metric distance in this case is given by:

$$ds^2 = dX^M dX^N \eta_{NM} = dx^m dx^n \eta_{nm} + \ell^2 (d\zeta^\mu d\zeta^{\bar{\nu}} \eta_{\bar{\nu}\mu} + d\zeta^{\bar{\mu}} d\zeta^\nu \eta_{\nu\bar{\mu}}) / 2$$

where $\eta_{\mu\bar{\nu}} = -\eta_{\bar{\nu}\mu} = \delta_\mu{}^\nu$ and η_{nm} is Minkowskian. The property coordinates ζ and $\bar{\zeta}$ are being taken as dimensionless, so a length scale ℓ has been introduced to ensure the metric distance carries the correct units. If we make a local non-abelian unitary property coordinate transformation of the form:

$$x' = x; \quad \zeta^{\mu'} = [e^{i\Theta(x)}]{}^{\mu\bar{\nu}} \zeta^{\bar{\nu}}; \quad \zeta^{\bar{\mu}'} = \zeta^{\bar{\nu}} [e^{-i\Theta(x)}]{}^{\nu\bar{\mu}}, \quad (1)$$

the above metric does *not* transform correctly as a tensor strictly should. To fix this problem it is essential to introduce non-abelian gauge fields $W_m{}^{\mu\bar{\nu}}$ (accompanied by a coupling constant e), through frame vectors as described in Appendix A.1. This

yields the following metric:

$$G_{MN} = \begin{pmatrix} g_{mn} + \frac{1}{2}e^2 l^2 \bar{\zeta}(W_m W_n + W_n W_m)\zeta - \frac{1}{2}iel^2(\bar{\zeta}W_m)^{\bar{\nu}} - \frac{1}{2}iel^2(W_m\zeta)^{\nu} & & \\ -\frac{1}{2}iel^2(\bar{\zeta}W_n)^{\bar{\mu}} & 0 & \frac{1}{2}l^2\delta^{\nu\bar{\mu}} \\ -\frac{1}{2}iel^2(W_n\zeta)^{\mu} & -\frac{1}{2}l^2\delta^{\mu\bar{\nu}} & 0. \end{pmatrix} \quad (2)$$

The reader may readily check, from the transformation properties of the supermetric under coordinate changes, that gauge covariance now follows naturally. In fact (2) is not the most general metric that transforms correctly under a local property phase transformation. Without including (ghost) fields that anticommute, which are likely to arise in a quantised version, the metric can be generalised by allowing for products of the property coordinates that are invariant under local phase transformations; these act like property curvature terms: $\zeta^{\bar{\mu}}\zeta^{\mu}$. This results in the following metric, with four independent curvature coefficients c_i :

$$G_{MN} = \begin{pmatrix} G_{mn} & G_{m\nu} & G_{m\bar{\nu}} \\ G_{\mu n} & 0 & G_{\mu\bar{\nu}} \\ G_{\bar{\mu} n} & G_{\bar{\mu}\nu} & 0 \end{pmatrix} \quad (3)$$

where

$$\begin{aligned} G_{mn} &= g_{mn}(1 + c_1\bar{\zeta}\zeta + c_2(\bar{\zeta}\zeta)^2) + \frac{1}{2}e^2 l^2 \bar{\zeta}(W_m W_n + W_n W_m)\zeta(1 + c_3\bar{\zeta}\zeta) \\ G_{m\nu} &= -\frac{1}{2}iel^2(\bar{\zeta}W_m)^{\bar{\nu}}(1 + c_3\bar{\zeta}\zeta) \\ G_{m\bar{\nu}} &= -\frac{1}{2}iel^2(W_m\zeta)^{\nu}(1 + c_3\bar{\zeta}\zeta) \\ G_{\mu\bar{\nu}} &= \frac{1}{2}l^2\delta^{\nu\bar{\mu}}(1 + c_3\bar{\zeta}\zeta + c_4(\bar{\zeta}\zeta)^2). \end{aligned} \quad (4)$$

The requirement that the metric transforms correctly as a tensor under local property phase transformations has severely restricted the nature of the metric. As can be seen from (4), there are only four free parameters introduced by incorporating as many U(2) invariants as possible into the metric^a. We envisage that the c_i 's are the expectation values of chargeless Higgs or dilaton fields. [Fermionic fields would carry a Lorentz spinor index which would conflict with Lorentz invariance if they made an appearance in the $G_{m\nu}$ sector.]

The subsequent formulae are simplified somewhat by scaling all the curvature coefficients to $c_3 \equiv c$ as follows, $c_1 = b_1c$, $c_2 = b_2c^2$, $c_4 = b_4c^2$, and by defining $b \equiv 2b_1 - b_2 + 2b_4 - 3$. In particular the superdeterminant of this metric can be found by considering eqn. 36 from our previous paper⁶ and reduced to

$$s \det(X) = \det(A - BD^{-1}C) \det(D)^{-1}. \quad (5)$$

^a If we want to separate the U(1) and SU(2) couplings, as may be required for electroweak theory, then we must replace $eW\zeta(1 + c\bar{\zeta}\zeta)$ by $e'W'I\zeta(1 + c'\bar{\zeta}\zeta) + e\underline{W}\cdot\underline{\sigma}\zeta(1 + c\bar{\zeta}\zeta)$. This complication is left for future work.

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Applying this to our metric results in the particular form,

$$\begin{aligned}\sqrt{-G_{..}} &= \frac{4}{l^4} \sqrt{-g} [1 + 2(c_1 - c_3)\bar{\zeta}\zeta + (c_1^2 + 2c_2 - 4c_1c_3 + 3c_3^2 - 2c_4)(\bar{\zeta}\zeta)^2] \\ &= \frac{4}{l^4} \sqrt{-g} [1 + 2(b_1 - 1)c\bar{\zeta}\zeta + (b_1^2 + 2b_2 - 4b_1 + 3 - 2b_4)(c\bar{\zeta}\zeta)^2].\end{aligned}\quad (6)$$

3. Ricci supertensor and superscalar curvature

Using eqn. 14 from our previous paper⁶, namely:

$$\Gamma_{MN}{}^K = [(-1)^{[M][N]+[L]}G_{ML,N} + (-1)^{[L]}G_{NL,M} - (-1)^{[L][M]+[L][N]+[L]}G_{MN,L}]G^{LK}/2, \quad (7)$$

and the metric tensor (4), we can evaluate the Christoffel symbols $\Gamma_{MN}{}^K$. These are listed in the Appendix A.3. From the Christoffel symbols the Ricci supertensor \mathcal{R}_{KM} can be found using eqn. 18 from our previous paper⁶:

$$\begin{aligned}\mathcal{R}^J{}_{KLM} &= (-1)^{[J]([K]+[L]+[M])} [(-1)^{[K][L]}\Gamma_{KM}{}^J{}_{,L} - (-1)^{[K][M]+[L][M]}\Gamma_{KL}{}^J{}_{,M} \\ &\quad + (-1)^{[L][M]}\Gamma_{KM}{}^R\Gamma_{RL}{}^J - \Gamma_{KL}{}^R\Gamma_{RM}{}^J],\end{aligned}\quad (8)$$

and then taking the trace to get

$$\mathcal{R}_{KM} = (-1)^{[K][L]}\mathcal{R}^L{}_{KLM}. \quad (9)$$

Finally the Ricci superscalar \mathcal{R} can be found by contracting with the metric to get

$$\mathcal{R} = G^{MK}\mathcal{R}_{KM}. \quad (10)$$

We are now in a position to evaluate the total Lagrangian density,

$$\mathcal{L} = \int d^2\zeta d^2\bar{\zeta} \sqrt{-G_{..}} \mathcal{R}. \quad (11)$$

The evaluation is aided by an algebraic computer program coded in Mathematica. This results in

$$\begin{aligned}\mathcal{L} &= \sqrt{-g_{..}} \left[\frac{8c^2b}{l^4} R^{[g]} - \frac{e^2c}{l^2} \text{Tr}(\mathcal{F}^{mn}\mathcal{F}_{mn}) \right. \\ &\quad \left. + \frac{16c^3}{l^6} (-24b_1b_2 + 38b_1^2 + 40b_2 - 110b_1 + 75 + 40b_1b_4 - 60b_4) \right],\end{aligned}\quad (12)$$

where $R^{[g]}$ is the gravitational space-time Ricci scalar and \mathcal{F}_{mn} is the non-abelian field tensor, $\mathcal{F}_{mn} = W_{n,m} - W_{m,n} - ie[W_m, W_n]$. We can now recognize

$$16\pi G_N \equiv \frac{e^2 l^2}{2bc}, \quad b = 2b_1 - b_2 + 2b_4 - 3; \quad (13)$$

$$\Lambda = c(24b_1b_2 - 38b_1^2 - 40b_2 + 110b_1 - 75 - 40b_1b_4 + 60b_4)/l^2b. \quad (14)$$

We see that the Yang-Mills Lagrangian arises naturally from the geometry, as well as a cosmological constant Λ . In our previous paper with one property coordinate we only had two free curvature parameters in the metric, which produced

a negative cosmological constant. Now with two property coordinates and the possibility of $(\bar{\zeta}\zeta)^2$ terms we have the opposite problem: there are four free curvature parameters and our cosmological constant is essentially unrestricted, though it is at least now consistent with a positive value^b. We envisage there will be additional symmetry considerations, when we examine reparametrizations of ζ or which arise from quantisation, that will restrict the possible values for the cosmological constant.

We can now look at variation of the Lagrangian density with respect to the space-time metric and the gauge field to produce field equations; this will provide a check that everything is working consistently at the level of the extended superspace X and with the Mathematica coding. The variation of the supermetric with respect to the space-time metric is simply just $\delta G_{mn} = (1 + c_1 \bar{\zeta}\zeta + c_2 (\bar{\zeta}\zeta)^2) \delta g_{mn}$. The variation of the Lagrangian density with respect to the space-time metric is then given by

$$\begin{aligned} & \int d^2\zeta d^2\bar{\zeta} \sqrt{-G_{..}} (R^{mn} - \frac{1}{2} G^{mn} R) \delta G_{mn} / \delta g_{mn} = \\ & \frac{16c^3}{l^6} (12b_1b_2 - 19b_1^2 - 20b_2 + 55b_1 - 75/2 - 20b_1b_4 + 30b_4) g^{mn} \\ & + \frac{8c^2b}{l^4} \left(R^{[g] mn} - \frac{1}{2} g^{mn} R^{[g]} \right) - \frac{2ce^2}{l^2} \text{Tr}(\mathcal{T}^{mn}), \end{aligned} \quad (15)$$

where $\mathcal{T}^{mn} = \mathcal{F}^{ml} \mathcal{F}^n_l - \frac{1}{4} g^{mn} \mathcal{F}^{lk} \mathcal{F}_{lk}$ is the Yang-Mills stress tensor. Equating this to zero results in the following field equations:

$$\begin{aligned} & R^{[g] mn} - \frac{1}{2} g^{mn} R^{[g]} - \frac{e^2 l^2}{4bc} \text{Tr}(\mathcal{T}^{mn}) \\ & + \frac{c}{l^2 b} (24b_1b_2 - 38b_1^2 - 40b_2 + 110b_1 - 75 - 40b_1b_4 + 60b_4) g^{mn} = 0, \end{aligned} \quad (16)$$

which is consistent with the Lagrangian in (12).

The next verification is the variation of the Lagrangian density with respect to the gauge field. First we express our gauge field in terms of a basis, $W_m^{\bar{\mu}\nu} = W_m^i \tau_i^{\bar{\mu}\nu}$, where $\underline{\tau} = (I, \underline{\sigma})$. We then consider the variation of our metric G_{MN} with respect to W_p^i .

$$\begin{aligned} \delta G_{mn} &= \frac{1}{2} e^2 l^2 \bar{\zeta} (\delta_m^p \tau_i W_n + \delta_n^p W_m \tau_i + \delta_n^p \tau_i W_m + \delta_m^p W_n \tau_i) \zeta (1 + c \bar{\zeta} \zeta) \delta W_p^i, \\ \delta G_{m\nu} &= -\frac{1}{2} e^2 (\bar{\zeta} \tau_i)^{\bar{\nu}} (1 + c \bar{\zeta} \zeta) \delta_m^p \delta W_p^i, \\ \delta G_{m\bar{\nu}} &= -\frac{1}{2} e^2 (\tau_i \zeta)^{\nu} (1 + c \bar{\zeta} \zeta) \delta_m^p \delta W_p^i. \end{aligned}$$

^bObserve that some simplification takes place if one chooses $b_2 = b_1^2$ and $b_4 = b_1$ but there is really no compelling reason for that choice at this stage.

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Next, we find that the variation of the Lagrangian density with respect to W_p^i does not involve curvatures coefficients other than $c_3 = c$:

$$\begin{aligned} & \int d^2\zeta d^2\bar{\zeta} \sqrt{-G_{..}} (R^{MN} - \frac{1}{2}G^{MN}R) \delta G_{NM} / \delta W^n \\ & \propto (W_{m,n} - W_{n,m})^{,m} + 2ie[W^m, W_{n,m}] + ie[W_{m,n}, W^m] \\ & + ie[W^m_{,m}, W_n] + e^2(W^m W_m W_n - 2W^m W_n W_m + W_n W^m W_m) \end{aligned} \quad (17)$$

Equating this to zero gives precisely the non abelian version of the Maxwell equations in free space, namely $\mathcal{D}^m \mathcal{F}_{mn} = 0$. This reassures us that the work is free of errors. Clearly adding curvature to space-time as well just gives the general relativistic versions of the equations of motion.

4. Source Superfields

To close the circle we need to ascertain that in our formalism scalar and spinor sources lead to the correct non-abelian interactions of the gauge fields with those sources. To that end it suffices to ignore all curvature constants except for $c_3 = c$, which enters the space-time property sector of the metric and accompanies eW . In this context we might regard attribute 1 as referring to neutrinicity and attribute 2 as referring to electricity to get an idea of the properties of the resulting expansions in property.

4.1. Scalars

To begin with ignore all gauging (and thus all curvatures in G). In that flat limit an anti-selfdual⁵ scalar superfield admits an expansion that produces a singlet Y and a triplet \underline{Z} of $SU(2)$:

$$\begin{aligned} \sqrt{2}\Phi &= Y[1 - (\bar{\zeta}\zeta)^2/2] + Z^0(\zeta^{\bar{1}}\zeta^1 - \zeta^{\bar{2}}\zeta^2) + Z^+\zeta^{\bar{1}}\zeta^2 + Z^-\zeta^{\bar{2}}\zeta^1 \\ &= Y[1 - (\bar{\zeta}\zeta)^2/2] + \bar{\zeta}\underline{Z}.\underline{\sigma}\zeta. \end{aligned} \quad (18)$$

Using the lemma $\int d^2\zeta d^2\bar{\zeta} (\bar{\zeta}A\zeta)(\bar{\zeta}B\zeta) = \text{Tr}(A)\text{Tr}(B) - \text{Tr}(AB)$, we readily establish the normalization,

$$\int d^2\zeta d^2\bar{\zeta} \Phi^2(X) = -(Y^2 + \underline{Z}.\underline{Z}). \quad (19)$$

Next we introduce curvature into the metric, but limited just to checking that the gauge interactions emerge correctly. Thus we discard all c_i except for $c_3 = c$, which multiplies eW ; in that case the Berezinian collapses to

$$\sqrt{G_{..}} \rightarrow 4[1 - 2c\bar{\zeta}\zeta + 3(c\bar{\zeta}\zeta)^2]/l^2$$

and the kinetic Lagrangian reduces to the property integral,

$$\begin{aligned} \int d^2\zeta d^2\bar{\zeta} \sqrt{G_{..}} G^{MN} \partial_N \Phi \partial_M \Phi &= -(1 - 3c^2) \partial^m Y \partial_m Y \\ &\quad - \text{Tr}[(\partial^m Z - ie[W^m, Z]).(\partial_m Z - ie[W_m, Z])] \end{aligned} \quad (20)$$

This is exactly what we would have expected for the interaction of the four scalar fields with the gauge field apart from a trivial renormalization of the Y -field.

4.2. Spinors

So far, so straightforward. However the fermion sources prove more difficult to handle. This is because the vielbein generalization for the ‘Dirac’ operator, viz. $\Gamma^A E_A^M \partial_M$ introduces a new set of ‘gamma-matrices’ Γ^α , which are fermionic in nature and should therefore obey the commutation rules, $[\Gamma^\alpha, \Gamma^{\bar{\beta}}] = 2I^{\alpha\bar{\beta}}$, instead of the usual Clifford ones. We show how to do this using a particular representation that entrains auxiliary anticommuting variables θ , in Appendix A.5. The net conclusion of that appendix is that we may discard the effect of the new set of Grassmannian gamma-matrices Γ^α , provided that Ψ is a singlet Θ of the new representation and carries that factor Θ .

Thus we take write the anti-selfdual fermionic superfield and its adjoint in flat space to be

$$\Psi = (\bar{\zeta}\psi + \psi^c\zeta)(1 - \bar{\zeta}\zeta)\Theta/2, \quad \bar{\Psi} = (-\bar{\psi}\zeta + \bar{\zeta}\bar{\psi}^c)(1 - \bar{\zeta}\zeta)\bar{\Theta}/2, \quad (21)$$

where $\bar{\zeta}\psi \equiv \zeta^1\psi^1 + \zeta^2\psi^2$ and ψ^c is the charge conjugate of ψ . In that way we check that after integrating over auxiliary θ , the mass and kinetic terms arise in the usual fashion:

$$\int d^2\zeta d^2\bar{\zeta} \bar{\Psi}\Psi = \int d^2\zeta d^2\bar{\zeta} (\bar{\zeta}\zeta)(1 - 2\bar{\zeta}\zeta)[\bar{\psi}\psi + \bar{\psi}^c\psi^c]/4 = -2\bar{\psi}\psi, \quad (22)$$

$$\int d^2\zeta d^2\bar{\zeta} \bar{\Psi}i\gamma.\partial\Psi = \int d^2\zeta d^2\bar{\zeta} (\bar{\zeta}\zeta)(1 - 2\bar{\zeta}\zeta)[\bar{\psi}i\gamma.\partial\psi + \bar{\psi}^ci\gamma.\partial\psi^c]/4 = -2\bar{\psi}i\gamma.\partial\psi. \quad (23)$$

We now proceed to curve the space through the frame vectors E , ignoring property curvature coefficients except for $c_3 = c$ and focussing on the gauge field which lives in the property-spacetime sector, viz.

$$E_A^M \partial_M = [1 + c\bar{\zeta}\zeta/2](e_a^m \partial_m + ie(W_a\zeta)^\mu \partial_\mu - ie(\bar{\zeta}W_a)^{\bar{\mu}} \partial_{\bar{\mu}} + '0').$$

The gauge field produces a derivative over the property, but also introduces a compensating property factor and the net result is that

$$-\int d^2\zeta d^2\bar{\zeta} (s \det E) \bar{\Psi} i \Gamma^A E_A^M \partial_M \Psi = \frac{8\sqrt{g..}}{l^4} (1 - \frac{c}{4}) [\bar{\psi}\gamma.(i\partial - eW)\psi + \bar{\psi}^c\gamma.(i\partial + eW)\psi^c]. \quad (24)$$

This is exactly what one would have anticipated and it confirms that the frame vector and resulting metric have precisely the right forms not only for scalar sources but for spinor ones too.

5. Conclusions

In this article we have proved, with two properties and a metric that possesses the correct structure in the property-spacetime sectors (for incorporating gauge transformations), that the Yang-Mills Lagrangian automatically follows upon integrating over the property sector. We have also shown that the interaction of the gauge field with scalar and spinor fields assumes the correct form. The inclusion of spacetime curvature merely makes the formalism x -coordinate independent and conform to standard GR.

What we have not done in this paper is treated the case where the spinor left and right chiralities behave differently, as in electroweak theory. This is left to for a subsequent paper because it brings in new concepts; it is technically quite difficult because one is dealing with four properties actually – two for the lepton and a further two for the neutrino – and has the added complication of large numbers of property invariants. Until we are able to systematise and simplify the number of c_i coefficients we must leave this subject in abeyance for now. However one point needs making before departing: it is that one has to pick the ce^2 for each gauge interaction to have a uniform value in order to guarantee gravitational universality; this becomes a significant constraint when the property symmetry group is a direct product of subgroups.

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Regular meetings with Dr Peter Jarvis have helped to illuminate a lot of issues.

Appendix A. Frame vectors, Inverse metric, Christoffel symbols and Ricci tensor components

Appendix A.1. Frame vectors

We will define the following to include property curvature in our frame vectors: $T = (1 + c_3\bar{\zeta}\zeta + c_4(\bar{\zeta}\zeta)^2)$ and $S = (1 + [c_1 - c_3]\bar{\zeta}\zeta + [c_2 - c_4 - c_3(c_1 - c_3)](\bar{\zeta}\zeta)^2)$. Note that $ST = (1 + c_1\bar{\zeta}\zeta + c_2(\bar{\zeta}\zeta)^2)$.

We adopt upper-triangular frame vectors of the form:

$$\mathcal{E}_M^A = \sqrt{T} \begin{pmatrix} \sqrt{S}e_m^a - ieW_m^{\alpha\bar{\nu}}\zeta^\nu & ie\zeta^{\bar{\nu}}W_m^{\nu\bar{\alpha}} \\ 0 & \delta_\mu^\alpha & 0 \\ 0 & 0 & \delta_{\bar{\mu}}^{\bar{\alpha}} \end{pmatrix} \quad (\text{A.1})$$

with inverse vielbein,

$$E_A^M = \frac{1}{\sqrt{T}} \begin{pmatrix} (\sqrt{S})^{-1}e_a^m & i(\sqrt{S})^{-1}eW_a^{\mu\bar{\nu}}\zeta^\nu & -i(\sqrt{S})^{-1}e\zeta^{\bar{\nu}}W_a^{\nu\bar{\mu}} \\ 0 & \delta_\alpha^\mu & 0 \\ 0 & 0 & \delta_{\bar{\alpha}}^{\bar{\mu}} \end{pmatrix}. \quad (\text{A.2})$$

These of course satisfy

$$\mathcal{E}_M^B E_B^N = \delta_M^N. \quad (\text{A.3})$$

The metric is produced via $G_{MN} = (-1)^{[A][N]}\mathcal{E}_M^A \mathcal{E}_N^B \mathcal{I}_{BA}$.

Appendix A.2. Inverse metric

The inverse metric is found in a similar fashion to the metric through

$$G^{MN} = (-1)^{[B][M]} \mathcal{I}^{BA} E_A{}^M E_B{}^N; \quad (\text{A.4})$$

this results in the following inverse metric:

$$G^{MN} = \frac{1}{ST} \begin{pmatrix} g^{mn} & ie(W^m \zeta)^\nu & -ie(\bar{\zeta} W^m)^{\bar{\nu}} \\ ie(W^n \zeta)^\mu & -e^2(W^k \zeta)^\mu (W_k \zeta)^\nu & \frac{2}{l^2} S \delta^{\mu\bar{\nu}} - e^2(\bar{\zeta} W^k)^{\bar{\nu}} (W_k \zeta)^\mu \\ -ie(\bar{\zeta} W^n)^{\bar{\mu}} - \frac{2}{l^2} S \delta^{\nu\bar{\mu}} + e^2(\bar{\zeta} W^k)^{\bar{\mu}} (W_k \zeta)^\nu & & -e^2(\bar{\zeta} W^k)^{\bar{\mu}} (\bar{\zeta} W_k)^{\bar{\nu}} \end{pmatrix}. \quad (\text{A.5})$$

Expanding out T and S give the following list of components of the inverse metric:

$$\begin{aligned} G^{mn} &= g^{mn} [1 - c_1 \bar{\zeta} \zeta + (c_1^2 - c_2)(\bar{\zeta} \zeta)^2], \\ G^{m\nu} &= ie(W^m \zeta)^\nu [1 - c_1 \bar{\zeta} \zeta], \\ G^{m\bar{\nu}} &= -ie(\bar{\zeta} W^m)^{\bar{\nu}} [1 - c_1 \bar{\zeta} \zeta], \\ G^{\mu\nu} &= -e^2(W^k \zeta)^\mu (W_k \zeta)^\nu [1 - c_1 \bar{\zeta} \zeta], \\ G^{\bar{\mu}\bar{\nu}} &= -e^2(\bar{\zeta} W^k)^{\bar{\mu}} (\bar{\zeta} W_k)^{\bar{\nu}} [1 - c_1 \bar{\zeta} \zeta], \\ G^{\mu\bar{\nu}} &= \frac{2}{l^2} \delta^{\mu\bar{\nu}} [1 - c_3 \bar{\zeta} \zeta + (c_3^2 - c_4)(\bar{\zeta} \zeta)^2] - e^2(\bar{\zeta} W^k)^{\bar{\nu}} (W_k \zeta)^\mu [1 - c_1 \bar{\zeta} \zeta]. \end{aligned}$$

Note that the metric and its inverse are still graded symmetric, $G_{MN} = (-1)^{[M][N]} G_{NM}$ and $G^{MN} = (-1)^{[M][N]} G^{NM}$. The $G^{\mu\nu}$ and $G^{\bar{\mu}\bar{\nu}}$ sectors of the inverse metric are no longer one by one like they were in the abelian case, so they can be non zero without breaking the symmetry.

Appendix A.3. Christoffel symbols

Using (7) and the metric from (4) we calculated the following list of Christoffel symbols. All of the algebra was assisted by use of Mathematica.

$$\Gamma_{mn}{}^l = \Gamma_{mn}^{[g]}{}^l - \frac{1}{4} e^2 l^2 g^{lk} (1 + (c_3 - c_1) \bar{\zeta} \zeta) \bar{\zeta} [W_m W_{n,k} - W_m W_{k,n} + W_{m,k} W_n - W_{k,m} W_n + ie W_m W_n W_k - ie W_k W_m W_n + (m \leftrightarrow n)] \zeta,$$

$$\begin{aligned} \Gamma_{mn}{}^\lambda &= ie \Gamma_{mn}^{[g]}{}^\lambda (W_k \zeta)^\lambda - \frac{1}{l^2} g_{mn} \zeta^\lambda [c_1 + (2c_2 - c_1 c_3) \bar{\zeta} \zeta] - \frac{1}{4} e^2 l^2 (W^k \zeta)^\lambda \bar{\zeta} [\\ &e W_k W_m W_n - e W_m W_n W_k + i W_m W_{n,k} - i W_m W_{k,n} + i W_{m,k} W_n - i W_{k,m} W_n + (m \leftrightarrow n)] \zeta - \frac{1}{2} c_3 e^2 \bar{\zeta} (W_m W_n + W_n W_m) \zeta \zeta^\lambda - \frac{1}{2} e [(e W_m W_n + i W_{m,n}) \zeta + (m \leftrightarrow n)]^\lambda, \end{aligned}$$

$$\begin{aligned} \Gamma_{mn}{}^{\bar{\lambda}} &= -ie \Gamma_{mn}^{[g]}{}^{\bar{\lambda}} (\bar{\zeta} W_k)^{\bar{\lambda}} - \frac{1}{l^2} g_{mn} \bar{\zeta}^{\bar{\lambda}} [c_1 + (2c_2 - c_1 c_3) \bar{\zeta} \zeta] + \frac{1}{4} e^2 l^2 (\bar{\zeta} W^k)^{\bar{\lambda}} \bar{\zeta} [\\ &(e W_k W_m W_n - e W_m W_n W_k + i W_m W_{n,k} - i W_m W_{k,n} + i W_{m,k} W_n - i W_{k,m} W_n + (m \leftrightarrow n)] \bar{\zeta} - \frac{1}{2} c_3 e^2 \bar{\zeta} (W_m W_n + W_n W_m) \bar{\zeta} \bar{\zeta}^{\bar{\lambda}} - \frac{1}{2} e [\bar{\zeta} (e W_m W_n - i W_{m,n}) + (m \leftrightarrow n)]^{\bar{\lambda}}, \end{aligned}$$

$$\begin{aligned} \Gamma_{m\nu}{}^l &= -\frac{1}{2} [c_1 + (2c_2 - c_1^2)(\bar{\zeta} \zeta)] \delta_m{}^l \zeta^{\bar{\nu}} \\ &- \frac{1}{4} e l^2 g^{lk} \bar{\zeta} [e W_m W_k - i W_{m,k} - (m \leftrightarrow k)]^{\bar{\nu}} [1 + (c_3 - c_1)(\bar{\zeta} \zeta)], \end{aligned}$$

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$$\begin{aligned}
\Gamma_{m\bar{\nu}}^l &= \frac{1}{2}\delta_m^l \zeta^\nu [c_1 + (2c_2 - c_1^2)(\bar{\zeta}\zeta)] \\
&\quad - \frac{1}{4}e l^2 g^{lk} \left(e[(W_m W_k - W_k W_m)\zeta]^\nu - i[(W_{m,k} - W_{k,m})\zeta]^\nu \right) [1 - (c_1 - c_3)(\bar{\zeta}\zeta)], \\
\Gamma_{m\nu}^\lambda &= \frac{i}{2}e[\bar{\zeta}W_m]^{\bar{\nu}} \zeta^\lambda c_3 (1 - c_3(\bar{\zeta}\zeta)) + \frac{i}{2}e\zeta^{\bar{\nu}} [W_m \zeta]^\lambda \left(c_3 - c_1 - (2c_2 + c_3^2 - c_1^2)(\bar{\zeta}\zeta) \right) \\
&\quad - ie(W_m)^{\lambda\bar{\nu}} \left(1 - c_4(\bar{\zeta}\zeta)^2 \right) + \frac{i}{4}e^2 l^2 [W^k \zeta]^\lambda \left(e[\bar{\zeta}(W_m W_k - W_k W_m)]^{\bar{\nu}} \right. \\
&\quad \left. - ie[\bar{\zeta}(W_{m,k} - W_{k,m})]^{\bar{\nu}} \right) \left(1 + (c_3 - c_1)(\bar{\zeta}\zeta) \right), \\
\Gamma_{m\bar{\nu}}^{\bar{\lambda}} &= -\frac{i}{2}e\zeta^{\bar{\lambda}} [W_m \zeta]^\nu c_3 [1 - c_3(\bar{\zeta}\zeta)] - \frac{i}{2}e[\bar{\zeta}W_m]^{\bar{\lambda}} \zeta^\nu \left(c_3 - c_1 - (2c_2 + c_3^2 - c_1^2)(\bar{\zeta}\zeta) \right) \\
&\quad + ie(W_m)^{\nu\bar{\lambda}} [1 - c_4(\bar{\zeta}\zeta)^2] - \frac{i}{4}e^2 l^2 [\bar{\zeta}W^k]^{\bar{\lambda}} \left(e[(W_m W_k - W_k W_m)\zeta]^\nu \right. \\
&\quad \left. - i[(W_{m,k} - W_{k,m})\zeta]^\nu \right) [1 + (c_3 - c_1)(\bar{\zeta}\zeta)], \\
\Gamma_{m\nu}^{\bar{\lambda}} &= -\frac{1}{4}e^2 l^2 [\bar{\zeta}W^k]^{\bar{\lambda}} \left([\bar{\zeta}(W_{m,k} - W_{k,m})]^{\bar{\nu}} + ie[\bar{\zeta}(W_m W_k - W_k W_m)]^{\bar{\nu}} \right) \\
&\quad + \frac{1}{2}ie \left(\zeta^{\bar{\nu}} [\bar{\zeta}W_m]^{\bar{\lambda}} (c_1 - c_3) - \zeta^{\bar{\lambda}} [\bar{\zeta}W_m]^{\bar{\nu}} c_3 \right), \\
\Gamma_{m\bar{\nu}}^\lambda &= \frac{1}{4}e^2 l^2 [W^k \zeta]^\lambda \left([(W_{m,k} - W_{k,m})\zeta]^\nu + ie[(W_m W_k - W_k W_m)\zeta]^\nu \right) \\
&\quad + \frac{1}{2}ie \left(\zeta^\nu [W_m \zeta]^\lambda (c_1 - c_3) - \zeta^\lambda [W_m \zeta]^\nu c_3 \right), \\
\Gamma_{\mu\nu}^\lambda &= -\frac{1}{2}\zeta^{\bar{\nu}} (c_3^2 - 2c_4)(\bar{\zeta}\zeta)\delta^{\lambda\bar{\mu}} - \frac{1}{2}\delta^{\lambda\bar{\nu}} \zeta^{\bar{\mu}} [c_3 - (c_3^2 - 2c_4)(\bar{\zeta}\zeta)] + \frac{1}{2}c_3 \delta^{\lambda\bar{\mu}} \zeta^{\bar{\nu}}, \\
\Gamma_{\bar{\mu}\bar{\nu}}^{\bar{\lambda}} &= \frac{1}{2}\zeta^\nu \delta^{\mu\bar{\lambda}} (c_3^2 - 2c_4)(\bar{\zeta}\zeta) + \frac{1}{2}\delta^{\nu\bar{\lambda}} \zeta^\mu [c_3 - (c_3^2 - 2c_4)(\bar{\zeta}\zeta)] - \frac{1}{2}c_3 \delta^{\mu\bar{\lambda}} \zeta^\nu, \\
\Gamma_{\mu\bar{\nu}}^l &= -\frac{i}{2}e l^2 c_4 \left(\zeta^{\bar{\mu}} [W^l \zeta]^\nu - [\bar{\zeta}W^l]^{\bar{\mu}} \zeta^\nu \right) (\bar{\zeta}\zeta), \\
\Gamma_{\mu\bar{\nu}}^\lambda &= \frac{1}{2}(c_3^2 - 2c_4)(\bar{\zeta}\zeta)\zeta^\nu \delta^{\lambda\bar{\mu}} + \frac{1}{2}(c_3^2 - 2c_4)(\bar{\zeta}\zeta)\zeta^\lambda \delta^{\nu\bar{\mu}} - \frac{1}{2}c_3(\delta^{\nu\bar{\mu}} \zeta^\lambda + \delta^{\lambda\bar{\mu}} \zeta^\nu), \\
\Gamma_{\mu\bar{\nu}}^{\bar{\lambda}} &= \frac{1}{2}(c_3^2 - 2c_4)(\bar{\zeta}\zeta)\zeta^{\bar{\mu}} \delta^{\nu\bar{\lambda}} + \frac{1}{2}(c_3^2 - 2c_4)(\bar{\zeta}\zeta)\zeta^{\bar{\lambda}} \delta^{\nu\bar{\mu}} - \frac{1}{2}c_3(\delta^{\nu\bar{\mu}} \zeta^{\bar{\lambda}} + \delta^{\nu\bar{\lambda}} \zeta^{\bar{\mu}}), \\
\Gamma_{\mu\nu}^l &= \Gamma_{\mu\nu}^{\bar{\lambda}} = \Gamma_{\bar{\mu}\bar{\nu}}^l = \Gamma_{\bar{\mu}\bar{\nu}}^\lambda = 0.
\end{aligned}$$

Appendix A.4. Ricci tensor components

Here we list the components of the Ricci tensor R^{MN} . To simplify these expressions we have ignored space-time curvature and set $c_1 = c_2 = c_4 = 0$ with $c_3 = c$. (Readers who wish to see the full set, including all other c_i should contact P.D.Stack directly.) Note that $F_{mn} = W_{n,m} - W_{m,n}$ and $\mathcal{F}_{mn} = F_{mn} - ie[W_m, W_n]$.

$$\begin{aligned}
R^{mn} = & -\frac{1}{4}e^2l^2[\bar{\zeta}(\mathcal{F}^m{}_k\mathcal{F}^{nk})\zeta] - c^2(\bar{\zeta}\zeta)[\bar{\zeta}(W^mW^n)\zeta] \\
& + \frac{1}{4}ce^3l^2([\bar{\zeta}W_k\zeta] - (\bar{\zeta}\zeta)\text{Tr}(W_k)[\bar{\zeta}(eW^kW^mW^n + eW^mW^nW^k - 2eW^mW^kW^n - \\
& i(F^{mk}W^n - W^mF^{nk}))\zeta] - e^2c^2([\bar{\zeta}W^m\zeta] - (\bar{\zeta}\zeta)\text{Tr}(W^m))[\bar{\zeta}W^n\zeta] \\
& - \frac{i}{4}ce^2l^2(\bar{\zeta}\zeta)[\bar{\zeta}(eF^m{}_kW^kW^n - eW^mW_kF^{nk} - i(F^m{}_kF^{nk} - e^2W_kW^mW^nW^k + \\
& e^2W^mW_kW^kW^n))\zeta] + (m \leftrightarrow n),
\end{aligned}$$

$$\begin{aligned}
R^{\mu\nu} = & -\frac{9c^2}{l^4}\zeta^\mu\zeta^\nu \\
& + \frac{1}{4}e^2[W_{k,m}\zeta]^\mu[(F^{km} - 2ie[W^k, W^m])\zeta]^\nu + \frac{1}{4}e^4[W_kW_m\zeta]^\mu[[W^k, W^m]\zeta]^\nu \\
& - \frac{i}{2}e^2[W_k\zeta]^\mu[(2eW^k{}_{,m}W^m - 2eW_mW^{k,m} - eW_m{}^{,k}W^m - eW_m{}^{,m}W^k + eW_mW^{m,k} + eW^kW_m{}^{,m} \\
& - 2ie^2W_mW^kW^m + ieW_m{}^{,k,m} - iW^k{}_{,m}{}^{,m} + ie^2W_mW^mW^k + ie^2W^kW_mW^m))\zeta]^\nu \\
& - (\mu \leftrightarrow \nu),
\end{aligned}$$

$$\begin{aligned}
R^{\bar{\mu}\bar{\nu}} = & -\frac{9c^2}{l^4}\bar{\zeta}^{\bar{\mu}}\bar{\zeta}^{\bar{\nu}} \\
& + \frac{1}{4}e^2[\bar{\zeta}W_{k,m}]^{\bar{\mu}}[\bar{\zeta}(F^{km} - 2ie[W^k, W^m])]^{\bar{\nu}} + \frac{1}{4}e^4[\bar{\zeta}W_kW_m]^{\bar{\mu}}[\bar{\zeta}[W^k, W^m]]^{\bar{\nu}} \\
& - \frac{i}{2}e^2[\bar{\zeta}W_k]^{\bar{\mu}}[\bar{\zeta}(2eW^k{}_{,m}W^m - 2eW_mW^{k,m} - eW_m{}^{,k}W^m - eW_m{}^{,m}W^k + eW_mW^{m,k} + eW^kW_m{}^{,m} \\
& - 2ie^2W_mW^kW^m + ieW_m{}^{,k,m} - iW^k{}_{,m}{}^{,m} + ie^2W_mW^mW^k + ie^2W^kW_mW^m))]^{\bar{\nu}} \\
& - (\bar{\mu} \leftrightarrow \bar{\nu}),
\end{aligned}$$

$$\begin{aligned}
R^{m\nu} = & \frac{1}{2}ce^2(\bar{\zeta}\zeta)\text{Tr}(W_k)[\mathcal{F}^{mk}\zeta]^\nu - \frac{1}{2}ce^2(\bar{\zeta}\zeta)[\mathcal{F}^m{}_kW^k\zeta]^\nu \\
& - ic^2e\frac{1}{l^2}(\bar{\zeta}\zeta)[W^m\zeta]^\nu - ic^2e\frac{1}{l^2}[\bar{\zeta}W^m\zeta]\zeta^\nu + ic^2e\frac{1}{l^2}(\bar{\zeta}\zeta)\text{Tr}(W^m)\zeta^\nu \\
& - \frac{1}{2}ce^2[\bar{\zeta}W_k\zeta][\mathcal{F}^{mk}\zeta]^\nu - \frac{i}{4}l^2e^3[\bar{\zeta}(\mathcal{F}^k{}_l\mathcal{F}^{ml} + \mathcal{F}^m{}_l\mathcal{F}^{kl})\zeta][W_k\zeta]^\nu \\
& - \frac{1}{2}e[(2eW^m{}_{,k}W^k - 2eW_kW^{m,k} - eW_k{}^{,k}W^m - eW_k{}^{,m}W^k + eW_kW^{k,m} + eW^mW_k{}^{,k} \\
& - 2ie^2W_kW^mW^k + iW_k{}^{,m,k} - iW^m{}_{,k}{}^{,k} + ie^2W_kW^kW^m + ie^2W^mW_kW^k)\zeta]^\nu,
\end{aligned}$$

$$\begin{aligned}
R^{m\bar{\nu}} = & \frac{1}{2}ce^2(\bar{\zeta}\zeta)\text{Tr}(W_k)[\bar{\zeta}\mathcal{F}^{mk}]^{\bar{\nu}} - \frac{1}{2}ce^2(\bar{\zeta}\zeta)[\bar{\zeta}W_k\mathcal{F}^{mk}]^{\bar{\nu}} \\
& + ic^2e\frac{1}{l^2}(\bar{\zeta}\zeta)[\bar{\zeta}W^m]^{\bar{\nu}} + ic^2e\frac{1}{l^2}[\bar{\zeta}W^m\zeta]\zeta^{\bar{\nu}} - ic^2e\frac{1}{l^2}(\bar{\zeta}\zeta)\text{Tr}(W^m)\zeta^{\bar{\nu}} \\
& - \frac{1}{2}ce^2[\bar{\zeta}W^k\zeta][\bar{\zeta}\mathcal{F}^{mk}]^{\bar{\nu}} + \frac{i}{4}l^2e^3[\bar{\zeta}(\mathcal{F}^k{}_l\mathcal{F}^{ml} + \mathcal{F}^m{}_l\mathcal{F}^{kl})\zeta][\bar{\zeta}W_k]^{\bar{\nu}} \\
& + \frac{1}{2}e[\bar{\zeta}(2eW^m{}_{,k}W^k - 2eW_kW^{m,k} - eW_k{}^{,k}W^m - eW_k{}^{,m}W^k + eW_kW^{k,m} + eW^mW_k{}^{,k} \\
& - 2ie^2W_kW^mW^k + iW_k{}^{,m,k} - iW^m{}_{,k}{}^{,k} + ie^2W_kW^kW^m + ie^2W^mW_kW^k))]^{\bar{\nu}},
\end{aligned}$$

$$\begin{aligned}
R^{\mu\bar{\nu}} = & \frac{1}{l^4}[20c - 44c^2(\bar{\zeta}\zeta) + 44c^3(\bar{\zeta}\zeta)^2]\delta^{\mu\bar{\nu}} + \frac{1}{l^4}[18c^2 - 48c^3(\bar{\zeta}\zeta)]\zeta^{\bar{\nu}}\zeta^\mu \\
& + \frac{1}{4}e^4l^2[\bar{\zeta}W_k]^{\bar{\nu}}[W_m\zeta]^\mu\bar{\zeta}(\mathcal{F}^k{}_l\mathcal{F}^{ml} + \mathcal{F}^m{}_l\mathcal{F}^{kl})\zeta \\
& + \frac{1}{2}e^2[\bar{\zeta}W_{k,m}]^{\bar{\nu}}[\mathcal{F}^{km}\zeta]^\mu + 2c^2e^2\frac{1}{l^2}(\bar{\zeta}\zeta)[\bar{\zeta}W_k]^{\bar{\nu}}[W^k\zeta]^\mu \\
& + c^2e^2\frac{1}{l^2}([\bar{\zeta}W^k]^{\bar{\nu}}\zeta^\mu + \zeta^{\bar{\nu}}[W_k\zeta]^\mu)([\bar{\zeta}W_k\zeta] - (\bar{\zeta}\zeta)\text{Tr}(W_k)) \\
& + \frac{i}{2}ce^3([\bar{\zeta}W_m]^{\bar{\nu}}[\mathcal{F}^{km}\zeta]^\mu - [\bar{\zeta}\mathcal{F}^{km}]^{\bar{\nu}}[W_m\zeta]^\mu)([\bar{\zeta}W_k\zeta] - (\bar{\zeta}\zeta)\text{Tr}(W_k)) \\
& + \frac{i}{2}ce^3(\bar{\zeta}\zeta)[\bar{\zeta}W_m\mathcal{F}^m{}_k]^{\bar{\nu}}[W^k\zeta]^\mu + \frac{i}{2}ce^3(\bar{\zeta}\zeta)[\mathcal{F}^k{}_mW^m\zeta]^\mu[\bar{\zeta}W_k]^{\bar{\nu}} \\
& - \frac{i}{2}e^3[\bar{\zeta}[W_k, W_m]]^{\bar{\nu}}[W^{k,m}\zeta]^\mu + \frac{1}{2}e^4[\bar{\zeta}W_kW_m]^{\bar{\nu}}[[W^k, W^m]\zeta]^\mu \\
& - \frac{i}{2}e^2[\bar{\zeta}W_k]^{\bar{\nu}}[(2eW^k{}_{,m}W^m - 2eW_mW^{k,m} - eW_m{}^{,k}W^m - eW_m{}^{,m}W^k + eW_mW^{m,k} + \\
& eW^kW_m{}^{,m} - 2ie^2W_mW^kW^m + iW_m{}^{,k,m} - iW^k{}_{,m}{}^{,m} + ie^2W_mW^mW^k + ie^2W^kW_mW^m)\zeta]^\mu \\
& + \frac{i}{2}e^2[W^k\zeta]^\mu[\bar{\zeta}(2eW_{k,m}W^m - 2eW_mW^{k,m} - eW_m{}^{,k}W^m - eW_m{}^{,m}W^k + eW_kW_m{}^{,m} \\
& + eW_mW^m{}_{,k} - 2ie^2W_mW_kW^m - iW_{k,m}{}^{,m} + iW_{m,k}{}^{,m} + ie^2W_kW_mW^m + ie^2W_mW^mW_k))]^{\bar{\nu}}.
\end{aligned}$$

Appendix A.5. *Square Rooting the Grassmann Metric*

Let $\Gamma^A P_A$ stand for the Dirac operator in flat space, where $P_A \equiv i\partial/\partial X^A$. The Dirac operator has to be bosonic like mass. Now $\Gamma^a \equiv \gamma^a$ and P_a which act in space-time are both bosonic, but P_α which serves as the Grassmann derivative is fermionic, so we require Γ^α to be fermionic too. Since we also demand that $(\Gamma^A P_A)^2 = \eta^{AB} P_B P_A$, we deduce that in the property sector,

$$[\Gamma^\alpha, \Gamma^\beta] = [\Gamma^{\bar{\alpha}}, \Gamma^{\bar{\beta}}] = 0, \quad [\Gamma^\alpha, \Gamma^{\bar{\beta}}] = 2\eta^{\alpha\bar{\beta}} = 2\delta_\beta^\alpha.$$

The first two commutators can be guaranteed by doubling the space and writing $\Gamma^\alpha = \sigma_+ \mathcal{O}^\alpha$, $\Gamma^{\bar{\alpha}} = \sigma_- \mathcal{O}^{\bar{\alpha}}$ since $\sigma_+^2 = \sigma_-^2 = 0$. This then leaves

$$[\Gamma^\alpha, \Gamma^{\bar{\beta}}] = \sigma_3 \{\mathcal{O}^\alpha, \mathcal{O}^{\bar{\beta}}\}/2 + [\mathcal{O}^\alpha, \mathcal{O}^{\bar{\beta}}]/2,$$

which must somehow be reduced to $2\delta_\beta^\alpha$.

Now these Γ^α or \mathcal{O}^α must carry the same number of attribute labels as $P_\alpha - 2$ for $U(2)$ – and be somehow distinguished from the original property coordinates ζ^α . One way to ensure this is to observe that if we introduce auxiliary a-scalars, $\Gamma^\alpha \equiv \theta^\alpha$ and $\Gamma^{\bar{\alpha}} \equiv \partial/\partial\theta^\alpha$, with

$$\{\theta^\alpha, \partial/\partial\theta^\beta\} = \delta_\beta^\alpha, \quad [\theta^\alpha, \partial/\partial\theta^\beta] = \delta_\beta^\alpha - 2(\partial/\partial\theta^\beta)\theta^\alpha$$

and make these act on *singlets* $\Theta = \theta^1\theta^2\ldots\theta^N$, we attain our goal. All told then $[\Gamma^\alpha, \Gamma^{\bar{\beta}}] = (1 + \sigma_3)\delta_\beta^\alpha$, when projected on to the singlet Θ . [There may be other ways of ‘square-rooting’ the Grassmann metric but our procedure more or less does the job.] We make use of this representation of Γ^α and $\Gamma^{\bar{\alpha}}$ in section 4.2. The net result is that if we multiply Ψ by Θ and integrate over auxiliary θ -space the part of the Dirac operator which involves Γ^α vanishes.

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